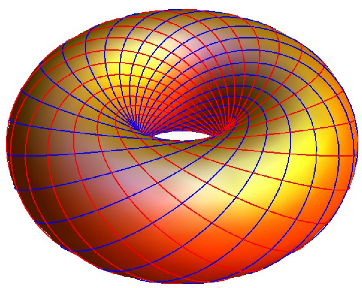
János Kollár

János Kollár spoke about classical and modern work on the classification of *celestial surfaces*. To a first approximation a celestial surface is a *cyclide*, a class of surfaces much studied in the 19th century. More precisely, Kollár defined a celestial surface to be a surface that contains at least two circles through a general point. Despite their definition having a metric character, celestial surfaces fall within the realm of algebraic geometry because it can be shown that a surface that contains at least two algebraic curves through a general point is locally algebraic.

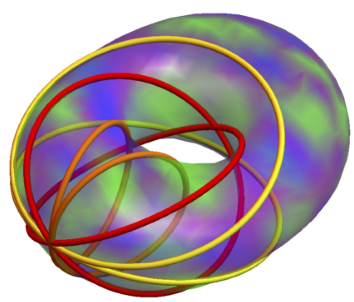
Basic examples are the plane, the sphere, and quadric surfaces—except that the two families of circles on a general quadric degenerate into a single family in the presence of circular symmetry. Kollár explained how the geometry in the quadric case can be understood by introducing the *ideal quadric* at infinity in $\mathbb{R}^n \subset \mathbb{P}^n$, defined by $x_0 = \sum_1^n x_i^2 = 0$, and by characterizing circles as conics that meet the ideal quadric in two conjugate point pairs.



A futher example is a torus formed from a circle as a surface of revolution. The method of construction immediately gives two familes of circles though every point. But there are also two other families that wind around the torus in opposing senses. They are attributed to Villarceau, who wrote about them in the mid 19th century, although they were known to the masons who built Strasbourg cathedral around 1300 AD.

In this example of a torus there are four families through each point. For a more general torus, there are in fact six: the reduction to four is a consequence of the special symmetry of a surface of revolution.

The underlying algebraic geometry is illustrated by starting with a quadric $Q \subset \mathbb{P}^3$ and blowing up four points. The resulting surface has 10 families of conics, but only six of these can be real. The circle families on the torus can be understood in a similar way by noting that, in algebraic-geometric terms, a torus is a degree four Del Pezzo surface with four double points. Other examples of surfaces with two families of circles have been exploited in architecture.



A general classification program was begun by Dupin, Kummer, and Darboux. Examples that were not known to them have been found in more recent years in all dimensions.

In 2001, Josef Schicho listed all the pairs (S, L) (S a normal projective surface, L an ample Cartier divisor) which give rise to surfaces embedded in higher dimensional spaces with at least two conics through each point. The way in which this works is illustrated in two particular cases:

- In the Veronese case $L = 2H$ and $\mathbb{P}^2$ is embedded in $\mathbb{P}^5$ by presenting $\mathbb{P}^5$ as the projective form of the six-dimensional space of homogeneous degree-two polynomials in three variables.
- In the double Segre case $L = (2, 2)$ and the embedding is $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ given by thinking of $\mathbb{P}^8$ as the projective space of homogeneous polynomials of degree two in two sets of variables.

Under these degree-two embeddings the images of lines are conics. In the second case, there are two conics through each point. In the first, there are infinitely many conics through each point. The key question is: can one choose the embedding more carefully so that these conics actually become circles?

Part of the answer is provided by applying a (genuinely) classical theorem that stereographic projection $S^n \to \mathbb{R}^n$ preserves circles. In consequence, there is an equivalence between (i) a circle contained in a surface in $\mathbb{R}^n$ and (ii) a conic contained in a surface in S^n, because a conic in S^n, being a plane section of the sphere, must be a circle. So the problem translates to the algebraic-geometric one of classifying real algebraic surfaces

$$F \subset S^n := (x_0^2 + \cdots + x_n^2 = 1) \subset \mathbb{R}^{n+1}$$

that admit at least two families of conics.

More generally, one has the problem of classifying the algebraic surfaces over a field k

$$F \subset Q^n := (q(x_0, \ldots, x_{n+1}) = 0) \subset \mathbb{P}_k^{n+1}$$

that admit at least two families of conics, where q is a quadratic. In the two particular cases above, it is the problem of describing the spaces

(1) $\mathrm{Map}_2(\mathbb{P}^2, Q^n)$—surfaces with a 2-dimensional family of circles and

(2) $\mathrm{Map}_2(\mathbb{P}^1 \times \mathbb{P}^1, Q^n)$—surfaces with two 1-dimensional families of circles,

where Map_2 denotes maps of degree two. It is generally hard to describe such spaces, but in these cases it reduces to a tractable algebraic problem. So, for example, if Q^4 is a smooth quadric 4-fold in $\mathbb{P}^5$ over $k = \bar{k}$, then $\mathrm{Map}_2(\mathbb{P}^2, Q^4)$ has five irreducible components, each birational to $\mathbb{P}^{20}$ and each understandable in terms of Veronese surfaces or quadruple planes.

The lecture concluded by considering the Veronese case over an arbitrary field. Here there is a one-to-one correspondence over any field between four-dimensional quadrics containing the Veronese surface and rational normal curves in $\mathbb{P}^4$. Again this can be formulated in entirely algebraic terms.

Images by courtesy of Niels Lubbes (first image) and Daniel Dreibelbis (second image).