

Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups of Real Rank One

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SUMMARY. Let G be a real semisimple Lie group. Harish-Chandra has defined the Schwartz space, $\mathcal{S}(G)$, on G . A tempered distribution on G is a continuous linear functional on $\mathcal{S}(G)$.

If the real rank of G equals one, Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [**3**(k), §24]. We restrict the Fourier transform map to $\mathcal{S}(G)$, and we compute the image of the space $\mathcal{S}(G)$ [Theorem 3]. This permits us to develop the theory of harmonic analysis for tempered distributions on G [Theorem 5].

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Bibliography

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§1. Introduction. Let G be a real semisimple Lie group. The Fourier transform map, \mathcal{F} , can be regarded as an isometry from $L^2(G)$ onto $L^2(\hat{G})$. $L^2(\hat{G})$ is a Hilbert space defined with the help of the discrete series, \mathcal{E}_d , and the various continuous series, \mathcal{E}_c , of irreducible unitary representations of G . $L^2(\hat{G})$ consists of certain functions whose domain is $\mathcal{E}_d \cup \mathcal{E}_c$, and whose range is the space of Hilbert-Schmidt operators on the Hilbert spaces on which the representations in $\mathcal{E}_d \cup \mathcal{E}_c$ act.

In [3(1)] Harish-Chandra introduces the Schwartz space, $\mathcal{E}(G)$, of functions on G . It is analogous to the space, $\mathcal{S}(\mathbf{R})$, of rapidly decreasing functions on the real line. $\mathcal{E}(G)$ is a Fréchet space. It is dense in $L^2(G)$, and its injection into $L^2(G)$ is continuous. It is of interest to ask about the image of $\mathcal{E}(G)$ in $L^2(\hat{G})$ under \mathcal{F} . There is a candidate, $\mathcal{E}(\hat{G})$, for this image space. $\mathcal{E}(\hat{G})$ is a Fréchet space defined by a natural family of seminorms on $L^2(\hat{G})$.

A tempered distribution on G is a continuous linear functional on $\mathcal{E}(G)$. If we can prove that the Fourier transform gives a topological isomorphism from $\mathcal{E}(G)$ onto $\mathcal{E}(\hat{G})$, we could define the Fourier transform of a tempered distribution as a continuous linear functional on $\mathcal{E}(\hat{G})$. This would include as a special case the theory of Fourier transforms on $L^2(G)$.

We confine ourselves to the case in which the real rank of G equals one. In this case Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [3(k), §24]. Our main result is Theorem 3, which asserts the bijectivity between $\mathcal{E}(G)$ and $\mathcal{E}(\hat{G})$ of the Fourier transform, \mathcal{F} .

The most difficult part of this theorem is to prove surjectivity. We have to show that the inverse Fourier transform of an element in $\mathcal{E}(\hat{G})$, which is *a priori* in $L^2(G)$, is actually in $\mathcal{E}(G)$. We use some estimates which Harish-Chandra develops from the study of a differential equation on G [3(1), §27]. In §9 we review his work and show that his estimates are actually uniform, in a sense which will become clear. In §10 we use these estimates to prove that $\mathcal{F}(\mathcal{E}(G))$ contains $\mathcal{E}_0(\hat{G})$, a subspace of $\mathcal{E}(\hat{G})$ associated with the discrete series.

To prove that $\mathcal{F}(\mathcal{E}(G))$ contains $\mathcal{E}_1(\hat{G})$, the subspace of $\mathcal{E}(\hat{G})$ associated with the continuous series, requires more work. It is necessary to derive a formula (Lemma 41) for the norms of certain linear transformations, $c^+(\Lambda)$ and $c^-(\Lambda)$, which arise in §12. This we do in §13 by studying a second-order symmetric differential operator on \mathfrak{a}_p , a one-dimensional subspace of the Lie algebra of G . As a byproduct of this formula we obtain in §14 a condition for irreducibility of certain representations in the continuous series.

For convenience we work with generalized spherical functions. We develop the pertinent information in §5 and then use it in §6 to prove the injectivity of the Fourier transform.

In §16 we define the Fourier transform of a tempered distribution on G . Theorem 6 proves that any continuous linear functional on $\mathcal{E}(\hat{G})$ is a certain sum of tempered distributions on the real line.

It seems likely that some of our methods can be used for proving the analogue of Theorem 3 for arbitrary G . The injectivity of the Fourier transform should

carry over quite easily. Harish-Chandra's estimates are proved in [3(1), §27] for arbitrary G . That these estimates are uniform can also be shown, although the proof of this is somewhat more complicated than in the real rank 1 case. Our proof of Lemma 27 does not carry over in general. However, it gives a good start toward a general proof.

The general Plancherel formula will be complicated by the existence of more than one continuous series of representations. However, in each continuous series linear transformations $c(\lambda)$ can be defined. The formulae in Lemma 41 can probably be proved, although perhaps not by our methods. In general, Lemma 44 would be proved by induction on the real rank of G . Harish-Chandra does this for ordinary spherical functions in [3(h), Theorem 3].

2. Preliminaries. Let G be a connected real semisimple Lie group with Lie algebra \mathfrak{g} . Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a fixed Cartan decomposition with Cartan involution θ . Let $\mathfrak{a}_{\mathfrak{p}}$ be a fixed maximal abelian subspace of \mathfrak{p} . The dimension of $\mathfrak{a}_{\mathfrak{p}}$ is called the real rank of G . We shall assume that $\dim \mathfrak{a}_{\mathfrak{p}} = 1$.

Let $\mathfrak{a}_{\mathfrak{k}}$ be a subspace of \mathfrak{k} such that

$$\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$$

is a Cartan subalgebra of \mathfrak{g} . Let K be the analytic subgroup of G corresponding to \mathfrak{k} . We assume that G has finite center. This implies that K is compact.

We can make further technical assumptions on G without losing generality. In order to do this we state some definitions of Harish-Chandra.

If L is a connected reductive Lie group over the reals, \mathbf{R} , with Lie algebra \mathfrak{l} , let

$$j: \mathfrak{l} \rightarrow \mathfrak{l}_{\mathbf{C}}$$

be inclusion into the complexification of \mathfrak{l} . (From now on, if \mathfrak{h} is any real Lie algebra we write $\mathfrak{h}_{\mathbf{C}}$ for its complexification.) Then if $L_{\mathbf{C}}$ is a complex analytic group with Lie algebra $\mathfrak{l}_{\mathbf{C}}$, $L_{\mathbf{C}}$ is called a complexification of L if j extends to a homomorphism of L into $L_{\mathbf{C}}$. Let $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{c}$, where \mathfrak{l}_1 is semisimple and \mathfrak{c} is abelian. Let $\mathfrak{l}_{1\mathbf{C}}$ and $\mathfrak{c}_{\mathbf{C}}$ be the respective complexifications of \mathfrak{l}_1 and \mathfrak{c} . Let $L_1, C(L_{1\mathbf{C}}, C_{\mathbf{C}})$ be the analytic subgroups of $L(L_{\mathbf{C}})$ corresponding to $\mathfrak{l}_1, \mathfrak{c}(\mathfrak{l}_{1\mathbf{C}}, \mathfrak{c}_{\mathbf{C}})$ respectively. We call $L_{\mathbf{C}}$ quasi-simply connected (q.s.c.) if $L_{1\mathbf{C}} \cap C_{\mathbf{C}} = \{1\}$ and if $L_{1\mathbf{C}}$ is simply connected. We say that L is q.s.c. if it has a q.s.c. complexification.

Fix a complexification $j: L \rightarrow L_{\mathbf{C}}$ and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{l} . Let A and $A_{\mathbf{C}}$ be the Cartan subgroups of L and $L_{\mathbf{C}}$ corresponding to \mathfrak{h} and $\mathfrak{h}_{\mathbf{C}}$ (that is, the centralizers of \mathfrak{h} and $\mathfrak{h}_{\mathbf{C}}$ in G and $G_{\mathbf{C}}$ respectively). Clearly $j(A) \subseteq A_{\mathbf{C}}$. It is known that $A_{\mathbf{C}}$ is connected [3(j), corollary to Lemma 27]. If λ is a linear functional on $\mathfrak{h}_{\mathbf{C}}$, there exists at most one complex analytic homomorphism

$$\xi_{\lambda}: A_{\mathbf{C}} \rightarrow \mathbf{C}$$

such that for every H in $\mathfrak{h}_{\mathbf{c}}$

$$\xi_{\lambda}(\exp H) = e^{\lambda(H)}.$$

We also write ξ_{λ} for the homomorphism

$$\xi_{\lambda} \circ j: A \rightarrow \mathbf{C}.$$

ξ_{λ} can be seen to be independent of the complexification $L_{\mathbf{c}}$ used, provided that ξ_{λ} is defined on that complexification.

Clearly ξ_{α} exists for any root α of $(\mathfrak{l}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$. If $P_{\mathfrak{h}}$ is the set of positive roots relative to some ordering, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P_{\mathfrak{h}}} \alpha.$$

It is easy to see that the question of the existence of ξ_{ρ} is independent of the ordering of the roots of $(\mathfrak{l}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$ and of the choice of Cartan subalgebra \mathfrak{h} . If ξ_{ρ} exists we call $L_{\mathbf{c}}$ acceptable. We say that L is acceptable if it has an acceptable complexification.

If $L_{\mathbf{c}}$ is q.s.c., it is known that it is acceptable [3(j), Lemma 29]. If $L_1 \cap C$ is finite, it is clear that L has a finite, and hence acceptable, cover.

Suppose L is a compact, connected acceptable Lie group with Lie algebra \mathfrak{l} . Let \mathfrak{h} , $P_{\mathfrak{h}}$, A , and ρ be defined as above. For each α define an element H_{α} in $\mathfrak{h}_{\mathbf{c}}$ by

$$B(H_{\alpha}, H) = \alpha(H)$$

for all H in $\mathfrak{h}_{\mathbf{c}}$, where B is the Killing form of $\mathfrak{h}_{\mathbf{c}}$ restricted to $\mathfrak{h}_{\mathbf{c}}$. Put

$$\tilde{\omega} = \prod_{\alpha \in P} H_{\alpha}.$$

$\tilde{\omega}$ is in $S(\mathfrak{h}_{\mathbf{c}})$, the symmetric algebra on $\mathfrak{h}_{\mathbf{c}}$. Let Π be the lattice of linear functionals

$$\lambda: (-1)^{1/2}\mathfrak{h} \rightarrow \mathbf{R}$$

for which ξ_{λ} exists. Let $\Pi' = \{\lambda \in \Pi: \tilde{\omega}(\lambda) \neq 0\}$. If W is the Weyl group of $(\mathfrak{l}_{\mathbf{c}}/\mathfrak{h}_{\mathbf{c}})$, W acts on $(-1)^{1/2}\mathfrak{h}$. Then W acts on Π as follows:

$$s\mu(H) = \mu(s^{-1}H)$$

for μ in Π , s in W , and H in $(-1)^{1/2}\mathfrak{h}$. For s in W , put $\varepsilon(s) = (-1)^{n(s)}$, where $n(s)$ is the number of positive roots that are mapped by s into negative roots. For h a regular element of A , put

$$\Delta(h) = \xi_{\rho}(h) \prod_{\alpha \in P_{\mathfrak{h}}} (1 - \xi_{\alpha}(h^{-1})).$$

LEMMA 1. *There is a map $\mu \rightarrow \sigma(\mu)$ from Π' onto the set of unitary equivalence classes of irreducible representations of L . $\sigma(\mu_1) = \sigma(\mu_2)$ if and only if $\mu_1 = s\mu_2$ for some s in W . Furthermore, if h is a regular element of A ,*

$$\text{tr } \sigma(\mu)(h) = (\text{sign } \tilde{\omega}(\mu)) \cdot \Delta(h)^{-1} \cdot \left(\sum_{s \in W} \varepsilon(s) \xi_{s\mu}(h) \right).$$

Also there exists a constant c_L , independent of μ , such that

$$\dim \sigma(\mu) = c_L |\tilde{\omega}(\mu)|.$$

Finally, if μ is in Π' , and $B(\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$, then $\mu - \rho$ is the highest weight of the representation of the Lie algebra \mathfrak{g} corresponding to $\sigma(\lambda)$.

PROOF. Since L has a finite q.s.c. cover, we will assume without loss of generality that L is q.s.c. We can assume further that L is semisimple. Then L is simply connected, so Π is precisely the lattice of weights of \mathfrak{h} [3(j), Lemma 29]. If μ' is a dominant integral function (in the terminology of [5, p. 215]), and if $\mu = \mu' + \rho$, then $B(\mu, \alpha) > 0$ for any α in $P_{\mathfrak{h}}$ so μ is in Π' . Conversely, if μ is in Π' , there exists a unique s such that $B(s\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$. Then $\mu' = \mu - \rho$ is a dominant integral function on \mathfrak{h} . This demonstrates the relation between μ and the highest weight of $\sigma(\mu)$. The correspondence between representations and dominant integral functions is well known (see [5, Chapter VII]).

The other two statements of the lemma follow from the Weyl character formula [5, p. 255] and the Weyl dimension formula [5, p. 257]. \square

Now let us return to our group G . By going to a finite cover we can assume that G is q.s.c. and hence acceptable. Thus, if $j: \mathfrak{g} \rightarrow \mathfrak{g}_c$ and G_c is a simply connected analytic group with Lie algebra \mathfrak{g}_c then j extends to a homomorphism

$$j: G \rightarrow G_c.$$

Now K is reductive. Therefore, by going to a further finite cover of G , we may also assume that K is acceptable.

If we understand the harmonic analysis of a finite cover, \tilde{G} , of G then we understand the theory for G . We merely throw out those unitary representations of \tilde{G} which are nontrivial on the kernel of the covering projection. Therefore, the above two assumptions can be made with no loss of generality.

There are two possibilities for G .

Case I. There exists a Cartan subalgebra \mathfrak{b} of \mathfrak{g} such that \mathfrak{b} is contained in \mathfrak{k} . We can assume that \mathfrak{b} has been chosen so that it contains \mathfrak{a}_+ . Then it is known that $\{\mathfrak{b}, \mathfrak{a}\}$ is a set of representatives of conjugacy classes of Cartan subalgebras of \mathfrak{g} .

Case II. Such a \mathfrak{b} does not exist. Then there is only one conjugacy class of Cartan subalgebras and it is represented by \mathfrak{a} .

We shall try as far as possible to deal with these two cases together. Whenever we speak of \mathfrak{b} , we shall be implicitly referring to Case I. However, any mention of \mathfrak{a} , unless otherwise stated, will refer to either case.

Let B be the Cartan subgroup of G corresponding to \mathfrak{b} . Since it is a maximal abelian subgroup in the compact connected Lie group K , it is connected [4, Corollary 2.7, p. 247].

Let A be the Cartan subgroup of G corresponding to \mathfrak{a} . Then

$$A = A_I A_p$$

where $A_p = \exp \mathfrak{a}_p$, and A_I is contained in K . In Case II, A_I is a Cartan subgroup of K and is connected. Otherwise, A_I may not be connected. In any case, let \mathfrak{m} and M be the centralizers of \mathfrak{a}_p in \mathfrak{k} and K , respectively. Then M is compact with a finite number of connected components.

Fix compatible orders on the real dual spaces of \mathfrak{a}_p and $\mathfrak{a}_p + (-1)^{1/2}\mathfrak{a}_t$. Let P be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$ relative to this order. Let P_+ be the set of roots in P which do not vanish on \mathfrak{a}_p and let P_M equal $P - P_+$. \mathfrak{a}_t is a Cartan subalgebra of the reductive Lie algebra \mathfrak{m} and we can regard P_M as the set of positive roots of $(\mathfrak{m}, \mathfrak{a}_t)$.

Let M^0 and A_I^0 be the connected components of M and A_I . Let W and W_1 be the Weyl groups of $(\mathfrak{g}/\mathfrak{a})$ and $(\mathfrak{m}/\mathfrak{a}_t)$, respectively. Now in any connected component of M , it is possible to choose an element γ_1 such that

$$\text{Ad } \gamma_1 \cdot \mathfrak{a}_t = \mathfrak{a}_t.$$

But $\text{Ad } \gamma_1$ leaves \mathfrak{a}_p pointwise fixed. Therefore, the action of γ_1 on \mathfrak{a}_t can be regarded as coming from an element of the subgroup of W generated by those roots in P which vanish on \mathfrak{a}_p . That is, the action of $\text{Ad } \gamma_1$ on \mathfrak{a}_t is the same as for some element in W_1 . Therefore, we can choose a new element γ , in the same component of M , that leaves \mathfrak{a}_t pointwise fixed. This means that γ is in A_I . Therefore, A_I has the same number of connected components as M .

As usual, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha, \quad \rho_M = \frac{1}{2} \sum_{\alpha \in P_M} \alpha.$$

Then since G is acceptable, it is known that M^0 is also acceptable and that for any a_1 in A_I^0

$$(2.1) \quad \xi_\rho(a_1) = \xi_{\rho_M}(a_1)$$

[see **3(j)**, Lemma 30].

Let \mathcal{E}_M be the set of equivalence classes of irreducible unitary representations of M . Let C be the set of irreducible characters of the group A_I (the set of characters coming from irreducible representations of A_I). For ζ in C and a in A_I write $\langle \zeta, a \rangle$ for the value of ζ at a . It is clear that W_1 operates on C .

Put $\tilde{\omega}^m = \prod_{\alpha \in P_M} H_\alpha$ and let L_1 be the lattice of real linear functionals, μ , on $(-1)^{1/2}\mathfrak{a}_t$ such that ξ_μ exists. Let $L'_1 = \{\mu \in L_1 : \tilde{\omega}^m(\mu) \neq 0\}$. Let $Z(A) = \{\gamma \in A_I : j(\gamma) \in \exp(-1)^{1/2}\mathfrak{a}_R\}$. Then $Z(A)$ is a finite subgroup of A_I . It is known that if γ is in $Z(A)$ and m is in M^0 , then γ and m commute [**3(j)**, Lemma 51]. Also $Z(A)A_I^0 = A_I$, by [**3(k)**, Lemma 20], so $Z(A)M^0 = M$. Let $Z(A)^0 = Z(A) \cap A_I^0$. $Z(A)^0$ is a central subgroup of both $Z(A)$ and M^0 . Then M is the central product of M^0 and $Z(A)$ with respect to $Z(A)^0$ (see [**2**, p. 29]). Thus if $\overline{M} = M^0 \times Z(A)$ and $\overline{Z(A)}^0 = \{(\gamma, \gamma^{-1}) : \gamma \in Z(A)^0\}$ then $\overline{Z(A)}^0$ is a discrete normal subgroup of \overline{M} . M is isomorphic to $\overline{M}/\overline{Z(A)}^0$. Similarly, if $\overline{A_I} = A_I^0 \times Z(A)$, then A_I is isomorphic to $\overline{A_I}/\overline{Z(A)}^0$. Therefore, irreducible representations of M (or A_I) are in one-to-one correspondence with representations of \overline{M} (or

A_I) which are trivial on $\overline{Z(A)}^0$. An irreducible representation of $\overline{A_I}$ is of the form $\xi_\mu \otimes \delta$, where μ is in L_1 and δ is an irreducible representation of $Z(A)$. Let C' be the set of irreducible characters ζ in C that come from representations $\xi_\mu \otimes \delta$ of \overline{M} for which μ is actually in L'_1 . If μ and ζ are so related, we shall write $\mu = \mu_\zeta$. We would like to prove a lemma which will relate the representations in \mathcal{E}_M with characters in C' .

Let σ be an arbitrary representation of M . Then

$$(2.2) \quad \sigma = \sigma_0 \times \varepsilon$$

where σ_0 and ε are irreducible representations of M^0 and $Z(A)$, respectively, such that for any γ_0 in $Z(A)^0$, $\sigma_0(\gamma_0) \otimes \varepsilon(\gamma_0^{-1})$ is the identity. $Z(A)^0$ is in the center of both M^0 and $Z(A)$ so $\sigma_0(\gamma_0)$ and $\varepsilon(\gamma_0)$ are both scalars. Therefore

$$\sigma_0(\gamma_0) = \varepsilon(\gamma_0).$$

Suppose that $\sigma_0 = \sigma_0(\mu)$ in the notation of Lemma 1. μ is a linear functional in L'_1 . Then there exists an s in W_1 such that $s\mu - \rho$ is the highest weight for σ_0 . Let γ_0 be an element in $Z(A)^0$. By looking at the action of $\xi_{s\mu} - \rho(\gamma_0)$ on a highest weight vector for σ_0 we see that the scalar $\sigma_0(\gamma_0)$ is equal to $\xi_{s\mu} - \rho(\gamma_0)$. Therefore

$$\varepsilon(\gamma_0) = \sigma_0(\gamma_0) = \xi_{s\mu}(\gamma_0)\xi_\rho(\gamma_0^{-1}) = \xi_\mu(s^{-1}\gamma_0)\xi_0(\gamma_0^{-1}).$$

However, γ_0 is in the center of M so $s^{-1}\gamma_0 = \gamma_0$. Therefore

$$(2.3) \quad \varepsilon(\gamma_0)\xi_\rho(\gamma_0) = \xi_\mu(\gamma_0)$$

for any γ_0 in $Z(A)^0$.

For any γ in $Z(A)$, define

$$(2.4) \quad \delta(\gamma) = \varepsilon(\gamma)\xi_\rho(\gamma).$$

This is an irreducible representation of $Z(A)$ and by (2.3), $\xi_\mu \otimes \delta$ can be regarded as an irreducible representation of A_I . Let

$$(2.5) \quad \langle \zeta, \gamma a_0 \rangle = \zeta_\mu(a_0) \cdot \text{tr } \delta(\gamma)$$

for a_0 in A_I^0 , γ in $Z(A)$. ζ is an element in C' and $\mu = \mu_\zeta$. Therefore, given a σ in \mathcal{E}_M , we have constructed an element ζ in C' . We write $\sigma = \sigma(\zeta)$.

Conversely, let us start with an element in C' . By working backward we can show that there is a unique element σ in \mathcal{E}_M such that $\sigma = \sigma(\zeta)$.

Suppose that a_0 is a regular element of A_I^0 and γ is in $Z(A)$. We wish to compute the trace of $\sigma(a_0\gamma)$. Define

$$\Delta_M(a_0) = \xi_\rho(a_0) \cdot \prod_{\alpha \in P_M} (1 - \xi_\alpha(a_0^{-1})).$$

In the above notation

$$\text{tr } \sigma(a_0\gamma) = \text{tr } \sigma_0(a_0) \cdot \text{tr } \varepsilon(\gamma).$$

But from Lemma 1,

$$\mathrm{tr} \sigma(a_0) = (\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \xi_{s\mu}(a_0) \right).$$

Therefore, the trace of $\sigma(a_0\gamma)$ is equal to

$$(\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \xi_\rho(\gamma^{-1}).$$

Now it is easy to show that if γ_c is in $j(Z(A))$ then $(\gamma_c)^2 = 1$. Therefore if γ is in $Z(A)$, $\xi_\rho(\gamma) = \xi_\rho(\gamma)^{-1}$. For future convenience, we rewrite the trace of $\sigma(a_0\gamma)$ as

$$(2.6) \quad (\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \xi_\rho(\gamma).$$

LEMMA 2. *There is a map $\zeta \rightarrow \sigma(\zeta)$ from C' onto \mathcal{E}_M . $\sigma(\zeta_1) = \sigma(\zeta_2)$ if and only if $s\zeta_1 = \zeta_2$ for some s in W_1 . If a_0 is a regular element in A_I^0 and γ is in $Z(A)$ then the trace of $\sigma(\zeta)(a_0\gamma)$ equals*

$$(\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \cdot \xi_\rho(\gamma).$$

Also, there exists a constant C_M , independent of ζ , such that

$$\dim \sigma(\zeta) = C_M \cdot |\tilde{\omega}^m(\mu_\zeta)| \cdot \dim \zeta.$$

($\dim \zeta$ means the dimension of the representation of A_I of which ζ is the character.)

PROOF. The dimension formula follows from Lemma 1. All other statements in the lemma follow from the above discussion. \square

Let us say that the linear functional μ_ζ is associated with σ if $\sigma = \sigma(\zeta)$, in the above notation. For any σ in \mathcal{E}_M there are exactly $[W_1]$ associated real linear functionals on \mathfrak{a}_+ .

Now with B there is associated a discrete series of unitary representations of G . With A there is associated a continuous series. We shall describe these.

For the discrete series there is a formal analogy with Lemma 1. Let Σ be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$ relative to some order. For any α in Σ define H_α in $(-1)^{1/2}\mathfrak{b}$ by the formula

$$B(H_\alpha, H) = \alpha(H)$$

for any H in \mathfrak{b}_c . Put $\tilde{\omega}^b = \prod_{\alpha \in \Sigma} H_\alpha$. Let L be the lattice of real linear functionals, λ , on $(-1)^{1/2}\mathfrak{b}$ such that ξ_λ exists. Let $L' = \{\lambda \in L: \tilde{\omega}^b(\lambda) \neq 0\}$. Let $N(B)$ be the normalizer of B in G . Define

$$W_G = W_{G,b} = N(B)/B.$$

This is a finite group. It acts on B and therefore on L .

An irreducible representation π of G on a Hilbert space \mathcal{H} is said to be square-integrable if there exist nonzero vectors Φ_1, Φ_2 in \mathcal{H} such that $(\Phi_1, \pi(x)\Phi_2)$ is a square-integrable function of x . If π and π' are square-integrable representations on \mathcal{H} and \mathcal{H}' and if π and π' are not unitarily equivalent, then for Φ_1, Φ_2 in \mathcal{H} and Φ'_1, Φ'_2 in \mathcal{H}' ,

$$(2.7) \quad \int_G (\Phi_1, \pi(x)\Phi_2)(\pi'(x)\Phi'_2, \Phi'_1) dx = 0.$$

On the other hand, there is a number $\beta(\pi)$, the formal degree of π , such that for every $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ and \mathcal{H} ,

$$(2.8) \quad \int_G (\Phi_1, \pi(x)\Phi_2)(\pi(x)\Psi_2, \Psi_1) dx = \beta(\pi)^{-1}(\Phi_1, \Psi_1)(\Psi_2, \Phi_2).$$

These are the Schur orthogonality relations on G . They are proved in [3(d), Theorem 1].

Let \mathcal{E}_d be the set of unitary equivalence classes of square-integrable representations of G . Harish-Chandra gives a map $\lambda \rightarrow \omega(\lambda)$ from L' onto \mathcal{E}_d [see 3(1), Theorem 16]. $\omega(\lambda_1) = \omega(\lambda_2)$ if and only if there is an s in W_G such that $s\lambda_1 = \lambda_2$. Finally, there is a constant C_G , independent of λ , such that

$$\beta(\omega(\lambda)) = C_G |\tilde{\omega}^b(\lambda)|.$$

LEMMA 3. $\{\beta(\omega) : \omega \in \mathcal{E}_d\}$ is bounded away from zero.

PROOF. It is clearly enough to show that for any α in Σ , $\{\lambda(H_\alpha)\}_{\lambda \in L'}$ is bounded away from zero. Let \tilde{L} be the lattice of real linear functionals on $(-1)^{1/2}\mathfrak{b}$ generated by the roots. Then it is known that L/\tilde{L} is isomorphic to the center of G , which is finite. It is also known that $\{\lambda(H_\alpha)\}_{\lambda \in \tilde{L}}$ is a lattice in \mathbf{R} . Therefore $\{\lambda(H_\alpha)\}_{\lambda \in L}$ is also a lattice in \mathbf{R} . But if λ is in L' , $\lambda(H_\alpha) \neq 0$, so the lemma follows. \square

Now we shall describe the continuous series. There is a linear functional μ_0 from \mathfrak{a}_p to \mathbf{R} such that the restriction of any root in P_+ to \mathfrak{a}_p is either μ_0 or $2\mu_0$. Fix H_0 in \mathfrak{a}_p so that $\mu_0(H_0) = 1$. Extend the definition of μ_0 to \mathfrak{a} by letting it equal zero on \mathfrak{a}_t .

Let $\mathfrak{n}_c = \sum_{\alpha \in P_+} \mathbf{C}X_\alpha$, where for any α in P , X_α is a fixed root vector. Let $\mathfrak{n} = \mathfrak{n}_c \cap \mathfrak{g}$. Let N be the analytic subgroup of G corresponding to \mathfrak{n} . It is well known (see [4, p. 373]) that the map

$$(k, a, n) \rightarrow kan, \quad k \in K, a \in A_p, n \in N,$$

is a diffeomorphism of $K \times A_p \times N$ with G . For f in $C_0^\infty(G)$,

$$(2.9) \quad \int_G f(x) dx = \int_{K \times A_p \times N} f(kan) e^{2\rho(\log a)} dk da dn$$

for a suitable normalization of the Haar measure dx . If $x = kan$, write $K(x) = k$ and $H(x) = \log a$.

It is clear that $P = MA_pN$ is a subgroup of G . If σ in \mathcal{E}_M acts on a finite dimensional Hilbert space V_σ , and if Λ is in \mathbf{R} , then the map σ_Λ from P into $\text{End}(V_\sigma)$ given by

$$\sigma_\Lambda(m \cdot \exp tH_0 \cdot n) = \sigma(m)e^{-i\Lambda t}, \quad m \in M, n \in N, t \in \mathbf{R}$$

is an irreducible unitary representation of P . (We shall sometimes write i instead of $(-1)^{1/2}$.) Let $\pi_{\sigma,\Lambda}$ be the unitary representation of G on the Hilbert space $\mathcal{H}_{\sigma,\Lambda}$ obtained by inducing σ_Λ from P to G .

Then $\mathcal{H}_{\sigma,\Lambda}$ is the set of functions Φ from G into V_σ such that

$$(2.10) \quad \Phi(x\xi^{-1}) = \sigma_\Lambda(\xi)\Phi(x), \quad x \in G, \xi \in P,$$

$$(2.11) \quad \Phi(k) \text{ is a Borel function on } K,$$

$$(2.12) \quad \int_K \|\Phi(k)\|^2 dk < \infty.$$

The inner product on $\mathcal{H}_{\sigma,\Lambda}$ is given by

$$(\Phi, \Psi) = \int_K (\Phi(k), \Psi(k))_{V_\sigma} dk, \quad \Phi, \Psi \in \mathcal{H}_{\sigma,\Lambda},$$

where $(\cdot, \cdot)_{V_\sigma}$ is the inner product in V_σ . If Φ is in $\mathcal{H}_{\sigma,\Lambda}$, $\pi_{\sigma,\Lambda}(y)\Phi$ is given by

$$(2.13) \quad (\pi_{\sigma,\Lambda}(y)\Phi)(x) = \Phi(y^{-1}x)e^{-\rho(H(y^{-1}x)) + \rho(H(x))}, \quad x, y \in G.$$

For any real Λ , and any Φ in $\mathcal{H}_{\sigma,\Lambda}$ we can define a function $\tilde{\Phi}$ from K to V_σ by restricting Φ to K . This identifies $\mathcal{H}_{\sigma,\Lambda}$ with a Hilbert space, \mathcal{H}_σ , of square-integrable functions from K into V_σ . \mathcal{H}_σ is independent of Λ . In fact, if π_σ is the representation of K obtained by inducing σ to K , \mathcal{H}_σ is the Hilbert space on which π_σ acts. The above equivalence between \mathcal{H}_σ and $\mathcal{H}_{\sigma,\Lambda}$ gives an intertwining operator between π_σ and $\pi_{\sigma,\Lambda}|_K$, the restriction of $\pi_{\sigma,\Lambda}$ to K .

Let M' be the normalizer of \mathfrak{a}_p in K . M is a normal subgroup of M' . M'/M is a group consisting of two elements, $\{1, \delta\}$ say. δ acts on \mathfrak{a}_p by reflection. δ also induces an automorphism of M , modulo the group of inner automorphisms. Therefore δ defines a bijection.

$$\delta: \sigma \rightarrow \sigma'$$

of \mathcal{E}_M onto itself. If we let δ act on P , we can transform the representation σ_Λ into the representation $(\sigma')_{-\Lambda}$. Now, if Λ is real and σ is in \mathcal{E}_M , it is known that the representation $\pi_{\sigma,\Lambda}$ is equivalent to $\pi_{\sigma',-\Lambda}$. Furthermore, the representations $\{\pi_{\sigma,\Lambda}\}$, $\sigma \in \mathcal{E}_M$, $\Lambda > 0$ are all irreducible and inequivalent [see 1, Theorem 7; 2].

For each σ in \mathcal{E}_M and $\Lambda \neq 0$, let $N_\sigma(\Lambda)$ be a fixed unitary intertwining operator between the representations $\pi_{\sigma,\Lambda}$ and $\pi_{\sigma',-\Lambda}$. Then

$$N_\sigma(\Lambda)\pi_{\sigma,\Lambda}(x)N_\sigma(\Lambda)^{-1} = \pi_{\sigma',-\Lambda}(x), \quad x \in G.$$

Notice that since $\pi_{\sigma,\Lambda}$ is irreducible,

$$(2.14) \quad N_{\sigma'}(-\Lambda) = N_\sigma(\Lambda)^{-1}.$$

It will be convenient to assign a positive real number to any equivalence class of representations in either \mathcal{E}_d or \mathcal{E}_M . If ω is in \mathcal{E}_d , choose λ in L' such that $\omega = \omega(\lambda)$. The Killing form, B , of $\mathfrak{g}_\mathbb{C}$ can be regarded as a positive definite form on either $(-1)^{1/2}\mathfrak{b}$ or its real dual space. Then put

$$|\omega|^2 = B(\lambda, \lambda).$$

Since W_G acts on $(-1)^{1/2}\mathfrak{b}$ as a group of isometries under the Killing form, $|\omega|$ is well defined. Similarly, for σ in \mathcal{E}_M , we define

$$|\sigma|^2 = B(\mu_\sigma, \mu_\sigma)$$

where μ_σ is any real linear functional on $(-1)^{1/2}\mathfrak{a}_\mathfrak{k}$ associated with σ . $|\sigma|$ is well defined by the above argument.

Let \mathcal{E}_K be the set of unitary equivalence classes of irreducible representations of K . Let \mathfrak{h} be the subspace of \mathfrak{k} which is equal to either \mathfrak{b} or $\mathfrak{a}_\mathfrak{k}$, depending on whether we are in Case I or Case II. \mathfrak{h} is a Cartan subalgebra of \mathfrak{k} . In either case, we have already ordered the dual space of \mathfrak{h} . K is acceptable by assumption, so the representations in \mathcal{E}_K can be indexed by certain real linear functionals on $(-1)^{1/2}\mathfrak{h}$ as in Lemma 1. If τ is in \mathcal{E}_K and $\tau = \tau(\mu)$ for some real linear functional μ on $(-1)^{1/2}\mathfrak{h}$, then we write

$$|\tau|^2 = B(\mu, \mu).$$

$|\tau|$ is well defined.

3. Plancherel formula for $L^2(G)$. In order to put the Plancherel formula for G into the form we want, we must discuss characters of unitary representations of G . To do this we must introduce some more notation of Harish-Chandra.

For t in \mathbf{R} , put $h_t = \exp tH_0$. For g in $C_0^\infty(M^0A_p)$, write

$$F_g^M(a_0h_t) = \Delta_M(a_0) \cdot \int_{M^0/A_I^0} g(m^{*-1}a_0h_tm^*) dm^*$$

for a_0 in A_I^0 and a_0h_t a regular element in A . Here dm^* is the invariant measure on the homogeneous space M^0/A_I^0 . It is known that there exists a constant $c_1 > 0$ such that for any g in $C_0^\infty(M^0A_p)$

$$(3.1) \quad \int_{M^0 \times \mathbf{R}} g(m_0h_t) dm_0 dt = c_1 \int_{A_I^0 \times \mathbf{R}} \overline{\Delta_M(a_0)} \cdot F_g^M(a_0h_t) da_0 dt$$

(see [3(j), Lemma 41]).

For a in A_I and ah_t a regular element in A write

$$\begin{aligned} \Delta(ah_t) &= \xi_\rho(ah_t) \cdot \prod_{\alpha \in P} (1 - \xi_\alpha(ah_t)^{-1}), \\ \varepsilon_{\mathbf{R}}(ah_t) &= 1 \quad \text{if } t > 0, \quad = -1 \quad \text{if } t < 0. \end{aligned}$$

If f is in $C_0^\infty(G)$ write

$$F_f(ah_t) = \varepsilon_{\mathbf{R}}(ah_t) \cdot \Delta(ah_t) \cdot \int_{G^*} f(x^{*-1}ah_tx^*) dx^*.$$

